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QUASI TRANSIENTS IN CLASS B AUDIO-FREQUENCY PUSH-PULL AMPLIFIERS*

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Summary—*Although class B audio-frequency amplifiers have been analyzed by many previous investigators, the effect of the leakage inductance of the output transformer or choke in producing quasi transients, i.e., exponential terms which recur periodically, has passed unnoticed. This paper gives equations for determining these quasi transients in the wave forms of the plate voltage, the plate current, and the output current, when the amplifier has reached a permanent state. The theoretical relations are derived from fundamental relations involving the tube characteristic, which is assumed to be linear, and the circuit external to the tubes. An equivalent circuit based on three-circuit transformer theory is also given to show the physical significance of the different terms in the equations. Cathode-ray oscillograms are presented in support of the theoretically calculated curves.*

Introduction

THE performance of push-pull class B audio-frequency amplifiers has been studied by several writers[1,2,3] recently either graphically or analytically. The graphical method has the advantage of being able to take into account the curvature of the tube characteristics while in the analytic method one has to assume linear tube characteristics in order to simplify algebraic relations. However, in both methods of study so far, no attempt has been made to consider the effect of the leakage inductances in the output transformer or coupling choke on the performance. True it is that in high quality output transformers[4] the leakage inductances between windings have been kept low, and that any trouble due to them has been more or less eliminated at the root by proper design, still it is not without practical value, aside from the theoretical interest of the problem, to be able to precalculate such effects, because when one has to make an economic choice of transformers in his equipment, it is quite important to know the limit one has to go to eliminate the leakage inductances.

Broadly speaking, the effects of leakage inductances in variable frequency applications, such as the output stage of an audio-frequency amplifier, are twofold. First, they cause the impedance of the circuit to vary with frequency, thus producing, on a constant potential input, a decreased output as frequency increases, and second, they introduce finite time constants into the circuit thus bringing into play transients which distort the wave as one of the tubes changes from a conducting condition to a blocking condition and vice versa. The first of these effects, viz., variable impedance, is common to all classes of audio-frequency amplifiers, whether class A or class B, and has been very amply treated by Terman[5] and his students. The second of these effects, viz., transient phenomena, is of importance, however, only in amplifiers, such as the class B, in which there is a periodic stoppage of current in the tubes. This second effect has probably been recognized by many to be one of the main troubles in the performance of class B audio-frequency amplifiers at high frequencies but, as stated before, it has not been studied analytically. In this paper, mathematical expressions for the plate voltage, the plate current, and the load current will be derived for class B push-pull audio-frequency amplifiers in terms of the tube constants, the load resistance, and the leakage reactances to take account of the latter's effects, and oscillograms will be presented to substantiate the theoretical relations.

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[1] L. E. Barton, Proc. I. R. E., vol. 19, pp. 1131–1149; July, (1931); vol. 20, pp. 1085–1100; July, (1932).

[2] J. R. Nelson, Proc. I.R.E., vol. 20, p. 1763; November, (1932); vol. 21, pp. 858–874; June, (1933).

[3] A. P-T. Sah, *Sci. Rep.*, Tsing Hua University, Peiping, China, series A, vol. 2, pp. 49–73, (1933).

[4] J. F. Peters, *Elect. Eng.*, vol. 55, pp. 34–35, (1936).

Assumptions and Steps in the Solution of the Problem

In order to avoid complicated analytic expressions, two assumptions will be made. The first is that the tube characteristic is linear. In other words, the plate resistance, ρ, and the amplification factor, μ, are taken to be constant, and when the tube is conducting, the following linear relation is assumed to hold:

$$i_p = \frac{e_p + \mu e_g}{\rho} \tag{1}$$

in which i_p, e_p, and e_g are instantaneous values of the plate current, the plate voltage, and the grid voltage, respectively. From (1) and the definition of a class B amplifier, the negative C bias voltage, E_c, will have a magnitude given by

$$E_c = \frac{E_b}{\mu} \tag{2}$$

where E_b is the voltage of the B source. As the purpose of this paper is to consider the quasi transients in the output stage of the amplifier, a second assumption made is that the peak input voltage E is equal to

[5] F. E. Terman, "Radio Engineering," McGraw-Hill Book Company.

the C bias so that the grid is not allowed to draw any current and produce additional transient effects.

Although the definition of an ideal class B amplifier requires that the current flow in one tube lasts exactly half of a cycle, the transients in the circuit will prolong this conducting period to more than a half cycle. It is thus obvious that even after a permanent state is reached, there will be periods at which both tubes in a class B push-pull amplifier are conducting. To facilitate the analysis, a complete cycle will therefore be divided into the following four periods (Fig. 1):

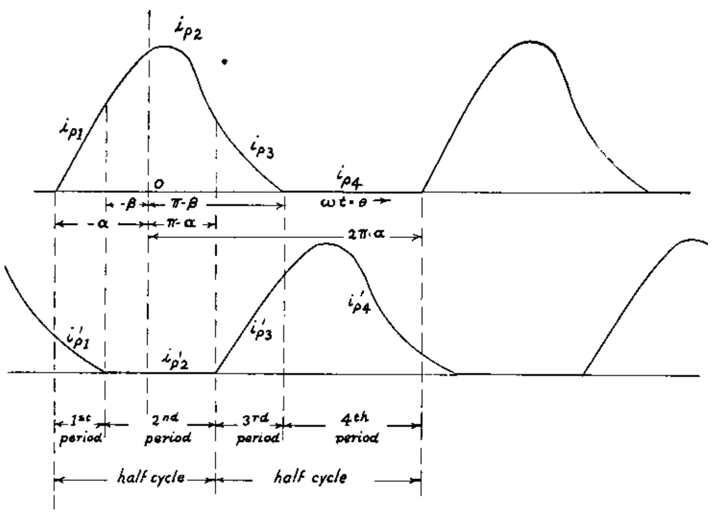


Fig. 1

(1) Both tubes are conducting with plate current in first tube increasing and that in second tube decreasing;

(2) First tube is conducting, but second tube is nonconducting;

(3) The two tubes are again both conducting with plate current in the first tube decreasing and that in the second increasing;

(4) First tube is nonconducting, but second tube is conducting.

To obtain general mathematical expressions for the different quantities in the different periods we need, in addition to the tube characteristic, viz., equation (1), the equations derivable from Kirchhoff's laws relating to the circuits external to the tube. With these written out they can be solved in the usual way.

The solution contains, of course, a steady-state term and one or two transients for each period. The constants of integration in the transient terms can be evaluated from the condition that at the transitional point from one period to the next the plate current is continuous. This condition of continuity of the plate current of one tube means also the continuity of all other quantities at the transitional points. In order to facilitate the algebraic work, it should be noted that the relations in the third and fourth periods will be exactly identical in value to those in the first and second periods, respectively, by changing all quantities referred to the first tube into those referring to the second tube, and vice versa. Also due to symmetry of the circuits, the first and second periods together will last one half cycle and the third and fourth periods the other half cycle, so that to specify the transitional points, only two auxiliary angles (say α and β) are necessary. Thus the first period will be taken to start at $\omega t = \theta = -\alpha$ and end at $\theta = -\beta$; the second period will last from $\theta = -\beta$ to $\theta = \pi - \alpha$; the third from $\theta = \pi - \alpha$ to $\theta = \pi - \beta$; and the fourth from $\theta = \pi - \beta$ to $\theta = 2\pi - \alpha$ (Fig. 1). In terms of these auxiliary variables α and β which can be found from two simultaneous equations involving the constants of the tube and the attached circuit, the constants of integration can be expressed; and when α and β are found, all the different parts of the complete wave form can be calculated and plotted.

Summarizing the above, one method of solving the problem will proceed as follows:

(1) Assume linear tube characteristics;

(2) Assume no grid current;

(3) From Kirchhoff's laws write down current and voltage relations for

case (a) when both tubes are conducting, and

case (b) when only one tube is conducting;

(4) Assume the open-circuit primary inductance of the transformer (i.e., with secondary open-circuited) to be very large; or in other words, neglect its effect and simplify the equations;

(5) Solve these equations simultaneously in the usual manner to obtain the general solution containing both "steady-state" and "transient" terms;

(6) Introduce two auxiliary angles α and β to specify the transition points from one period to the next;

(7) Impose the condition of continuity at the transition points and obtain two simultaneous equations involving α and β and express constants of integration in terms of α and β;

(8) Find α and β by trial and error;

(9) Evaluate the constants of integration from the values of α and β;

(10) Calculate the wave forms point by point from the equations obtained in step (5).

The General Solution

When the general circuit as shown in Fig. 2 is considered, it will be shown in the Appendix that after the effect of the primary induct-

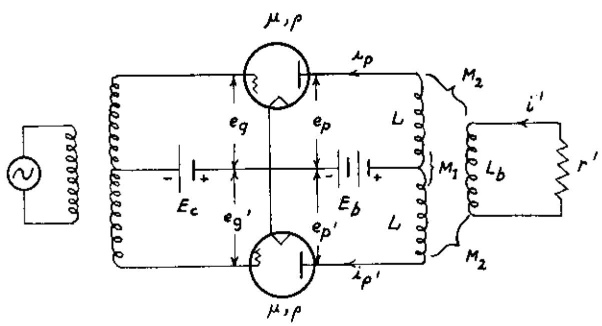


Fig. 2.—$r=(L/L_b)r'$; $i=\sqrt{L_b/L}\;i'$; $e_g=-E_c+E\cos\theta$; $e_g'=-E_c-E\cos\theta$; $L_2=L-M_2^2/L_b$; $L_1=L-M_1$.

ance L is neglected, the general solutions for the various quantities are: (a) When both tubes are conducting,

$$\left.\begin{aligned}
i_p &= C\epsilon^{-s\theta}+B\epsilon^{-m\theta}+\frac{\mu E\cos(\theta-\cot^{-1}s)}{Z_s}\\
i_p' &= -C\epsilon^{-s\theta}+B\epsilon^{-m\theta}-\frac{\mu E\cos(\theta-\cot^{-1}s)}{Z_s}\\
e_p &= E_b+\rho C\epsilon^{-s\theta}+\rho B\epsilon^{-m\theta}-\frac{Z_1\mu E\cos(\theta-\cot^{-1}s+\phi_1)}{Z_s}\\
e_p' &= E_b-\rho C\epsilon^{-s\theta}+\rho B\epsilon^{-m\theta}+\frac{Z_1\mu E\cos(\theta-\cot^{-1}s+\phi_1)}{Z_s}\\
i &= 2C\epsilon^{-s\theta}+\frac{2\mu E\cos(\theta-\cot^{-1}s)}{Z_s}
\end{aligned}\right\}\quad(3)$$

wherein,

$$s=\frac{2r+\rho}{\omega(2L_2-L_1)};\qquad m=\frac{\rho}{\omega L_1};$$

$$Z_s^2=(2r+\rho)^2+\omega^2(2L_2-L_1)^2;\qquad Z_1^2=4r^2+\omega^2(2L_2-L_1)^2;$$

$\cos\phi_1=\dfrac{2r}{Z_1}$; and B and C are constants of integration;

and (b) when one tube is conducting while the other is not, then

$$\left.\begin{aligned}
i_p &= A\epsilon^{-n\theta}+\frac{\mu E\cos(\theta-\cot^{-1}n)}{Z_n},\\
i_p' &= 0,\\
e_p &= E_b+\rho A\epsilon^{-n\theta}-\frac{Z_2\mu E\cos(\theta-\cot^{-1}n+\phi_2)}{Z_n},\\
e_p' &= E_b+\frac{(r+\rho)L_1-\rho L_2}{L_2}A\epsilon^{-n\theta}+\frac{Z_3\mu E\cos(\theta-\cot^{-1}n+\phi_3)}{Z_n},\\
i &= A\epsilon^{-n\theta}+\frac{\mu E\cos(\theta-\cot^{-1}n)}{Z_n},
\end{aligned}\right\}\quad(4)$$

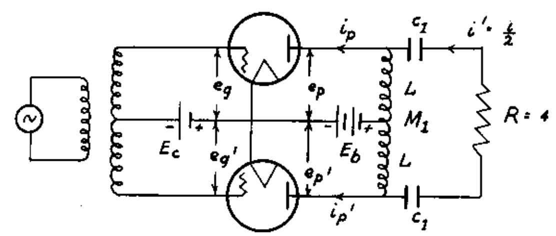


Fig. 3

wherein,

$$n=\frac{r+\rho}{\omega L_2};$$

$$Z_n^2=(r+\rho)^2+\omega^2L_2^2;\; Z_2^2=r^2+\omega^2L_2^2;\; Z_3^2=r^2+\omega^2(L_2-L_1)^2;$$

$\cos\phi_2=\dfrac{r}{Z_2}$; $\cos\phi_3=\dfrac{r}{Z_3}$; and

A is a third constant of integration. In case the secondary is completely coupled to the primaries without leakage, which is realized in practice by connecting the load resistance $r'=R=4r$ across the plates through large insulating condensers C_1 as shown in Fig. 3 then $2L_2=L_1$ and $C=0$, and the equations (3) simplify to

$$\left.\begin{aligned} i_p &= B\epsilon^{-m\theta} + \frac{\mu E \cos\theta}{2r+\rho}, \\ i_p' &= B\epsilon^{-m\theta} - \frac{\mu E \cos\theta}{2r+\rho} \\ e_p &= E_b + \rho B\epsilon^{-m\theta} - \frac{2r\mu E\cos\theta}{2r+\rho}, \\ e_p' &= E_b + \rho B\epsilon^{-m\theta} + \frac{2r\mu E\cos\theta}{2r+\rho}, \\ i &= \frac{2\mu E\cos\theta}{2r+\rho}; \end{aligned}\right\} \tag{5}$$

while, e_p' of (4) becomes

$$e_p' = E_b + (2r+\rho)A\epsilon^{-n\theta} + \frac{Z_2\mu E\cos(\theta - \cot^{-1} n - \phi_2)}{Z_n}. \tag{4a}$$

If subscripts 1, 2, 3, and 4 are used to distinguish the four periods as shown in Fig. 1 and unprimed and primed symbols to refer to the two tubes, then due to symmetry of arrangement

$$\left.\begin{aligned} i_{p3}(\theta) &= i_{p1}'(\theta-\pi) \\ i_{p3}'(\theta) &= i_{p1}(\theta-\pi) \\ e_{p3}(\theta) &= e_{p1}'(\theta-\pi) \\ e_{p3}'(\theta) &= e_{p1}(\theta-\pi) \\ i_3(\theta) &= -\,i_1(\theta-\pi) \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} i_{p4}(\theta) &= i_{p2}'(\theta-\pi) \\ i_{p4}'(\theta) &= i_{p2}(\theta-\pi) \\ e_{p4}(\theta) &= e_{p2}'(\theta-\pi) \\ e_{p4}'(\theta) &= e_{p2}(\theta-\pi) \\ i_4(\theta) &= -\,i_2(\theta-\pi). \end{aligned}\right\} \tag{7}$$

It is thus seen that only the two sets of general solutions as given in (3) and (4) are needed. Each one of the left-hand symbols in the set of (3) or (5) may then be given a subscript 1 to denote their respective values in the first period and those in set (4) a subscript 2 to denote values in the second period. The values in the third and fourth periods are obtained from these two sets through the relations (6) and (7).

Equivalent Circuit

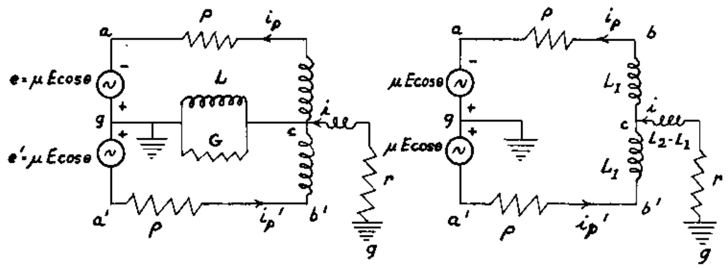
The solution given in detail in the Appendix is built up from fundamental relations. It is a little bit laborious and for practical analysis, perhaps, one would like to know whether it is possible to devise an equivalent circuit from which the above solutions may be obtained. Recognizing the fact that the output transformer in a push-pull circuit is nothing but a three-circuit transformer, the equivalent circuit for the latter properly combined with the equivalent circuit for the tubes should, from our physical intuition, give us the correct solution. Indeed the following equivalent circuit, Fig. 4, will be found to describe the push-pull amplifier exactly. Two points about the circuit may be mentioned. The first is that due to the differential effect of the currents i_p and i_p' in the push-pull arrangement, their positive directions are as shown while the equivalent voltage e' acting on the primed tube will have a positive direction opposite to that of i_p' to be in conformity with the actual push-pull arrangement. The fact that in the actual circuit diagram the positive directions of i_p and i_p' have been drawn away from their junction and that the grid voltages are 180 degrees out of phase must not misguide one to infer that in the equivalent circuit diagram the positive directions of i_p' and e' would be opposite to what are shown in Fig. 4. The second point to note is that all the resistances and inductances are referred to one tube only; in other words, all values connected across the tubes from plate to plate should be divided by four. Thus $2L_1$ is the total leakage inductance measured across one half of the primary windings when the other half is short-circuited and the secondary open-circuited, while L_2 is the total leakage inductance measured also across one half of the primary winding (i.e., from outer terminal to center tap) with secondary short-circuited and the other half of the primary open-circuited. It should be noted again that when the secondary is completely coupled to the whole primary without leakage, $L_2=L_1/2$ and there is, in fact, a negative value of inductance (i.e., $-L_1/2$) existing in series with the load resistance r under such a condition.

Time Constants and Impedances Based on Equivalent Circuit

The theory of a three-circuit transformer is now quite well known and is already available in many textbooks.[6] We shall, there-

[6] O. G. C. Dahl, "Electric Circuits," vol. 1, McGraw-Hill Book Company.

fore, not deduce the equivalent circuit but rather show that the results calculated from the equivalent circuit give the correct time constants and steady-state impedance in the two cases of (a) both tubes conducting and (b) only one tube conducting. We shall assume the exciting admittance of the transformer, i.e., the parallel branch denoted by L and G to be nonexistent. Thus when both tubes are conducting, there are three meshes to consider, Fig. 4(b); viz., (1) $abcb'a'g$; (2) $gabcg$, and (3) $ga'b'cg$. Since the circuit is symmetrical the last two meshes have identical constants, so that there are only two time constants. From the mesh $abcb'a'$, the total resistance in series is 2ρ and the total inductance $2L_1$, giving a time constant L_1/ρ, or an equivalent Q, if that term may be used, of $\omega L_1/\rho$. The transient current with



(a) (b)

Fig. 4

this time constant appears, however, only in the plate currents i_p and i_p'. It gives rise to the term

$$B\epsilon^{-m\theta} = B\epsilon^{-(\rho/L_1)t} \tag{8}$$

of (3) or (5). The second time constant is found from the mesh $gabcg$ with the branch $gabc$ in parallel with $ga'b'c$. This contains a total resistance $r+\rho/2$ and a total inductance $L_2-L_1/2$. Hence the time constant is $(2L_2-L_1)/(2r+\rho)$. The transient term having this time constant is associated with all three currents, i, i_p, and i_p'. If it is to be taken as positive in case of i and i_p, it should be negative in case of i_p' due to the assumed positive direction of currents. This is then responsible for the term (equation (3)),

$$C\epsilon^{-s\theta} = C\epsilon^{-(2r+\rho)t/(2L_2-L_1)} \tag{9}$$

in i and i_p and its negative in i_p'.

Except for the transient term $B\epsilon^{-m\theta}$ there is no steady-state current in the mesh $abcb'a'ga$. The steady-state current in the load r is divided equally between i_p and i_p'. The impedance to the flow of this current i in the load r is, in complex notation,

$$\dot{Z} = r + \frac{\rho}{2} + \frac{j\omega(2L_2 - L_1)}{2} \tag{10}$$

which accounts for the steady-state terms given at the end of each one of the equations (3) or (5), the negative sign of this term in i_p' being again due to the fact that the positive direction of i_p is what is shown. All the currents i_p, i_p', and i are thus seen correctly obtained from the equivalent circuit.

It is of interest here to note that when all constants are referred not to one half of the primary but to plate to plate, then the above impedance becomes:

$$\dot{Z}_{pp} = 4\dot{Z} = 4r + 2\rho + 2j\omega(2L_2 - L_1) = R + 2\rho + 2j\omega(2L_2 - L_1) \tag{11}$$

by setting $R=4r$, and the voltage to produce the current is $2\mu E \cos\theta$, giving a steady-state current

$$\dot{I}_{pp} = \frac{2\mu\dot{E}}{R + 2\rho + 2j\omega(2L_2 - L_1)} \tag{12}$$

showing that the circuit may also be assumed to be a series circuit as shown in Fig. 5, where the tubes are considered truly in series, for the applied electromotive forces are adding and the total internal plate

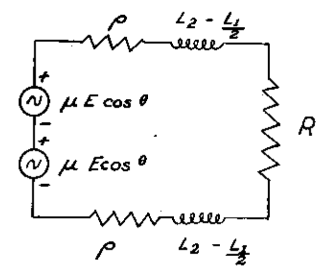


Fig. 5

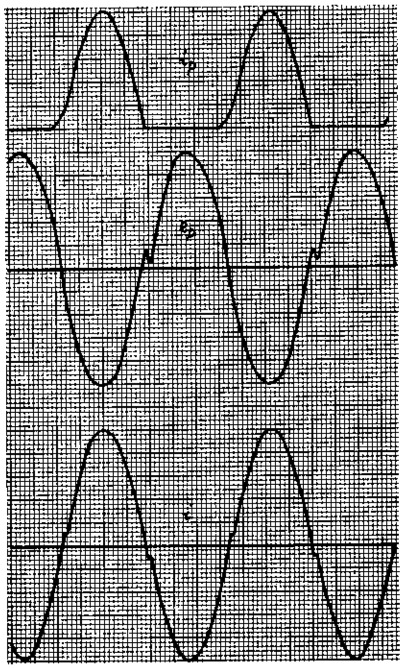
resistance is 2ρ. This equivalent circuit is just as good as the one given in Fig. 4, provided one is not interested in the transient terms, for instance in case of class A amplifiers. However, if the transient terms are to be evaluated, the "series equivalent circuit" given in Fig. 5 has to be replaced by the "parallel" arrangement of Fig. 4 in order to get correct results.

Coming next to the second period of operation in which one tube is conducting alone, the mesh $gab'c$ is simply considered open-circuited in finding the impedance; i.e.,

$$\dot{Z}_n = \rho + r + j\omega L_2 \tag{13}$$

and the time constant is $L_2/(\rho+r)$ or equivalent Q is $\omega L_2/(\rho+r)$. The transient and the steady-state terms in i and i_p found in (4) exactly correspond with these considerations.

The equivalent circuit is not very well adapted to the evaluation of the plate voltages on the tubes. To find these, simply note that in case



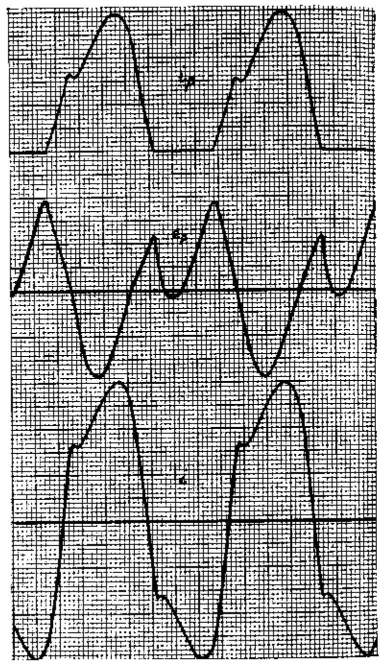


Fig. 6—$r=\rho=4\omega L_1$; $2L_2=L_1$; $m=4$; $n=16$.
Fig. 7—$r=\rho/2=\omega L_1/4$; $2L_2=L_1$; $m=1/2$; $n=3/2$.

i_p and i_p' are both positive, the linear characteristic of the tube requires

$$e_p = \rho i_p - \mu e_g = E_b - \mu E \cos\theta + \rho i_p$$

and,

$$e_p' = \rho i_p' - \mu e_g' = E_b + \mu E \cos\theta + \rho i_p' \qquad (14)$$

and that in case $i_p=0$, due to the blocking of the tube,

$$e_p' = E_b + [r + p(L_2 - L_1)]i_p. \qquad (15)$$

Some Calculated Theoretical Curves

To show how the transient terms affect the wave shapes of the different quantities, two cases showing extreme conditions are given here, assuming the output coupling to be a choke; i.e., $2L_2=L_1$. The circuit parameters for these two cases are as follows:

(A) $$r = \rho = 4\omega L_1$$

and,

(B) $$2r = \rho = 1/2\omega L_1.$$

They are plotted in Figs. 6 and 7. The following points in the curves may be noted:

(1) Plate currents flowing more than 180 degrees, the angle of current flow in case B is as much as 230 degrees;

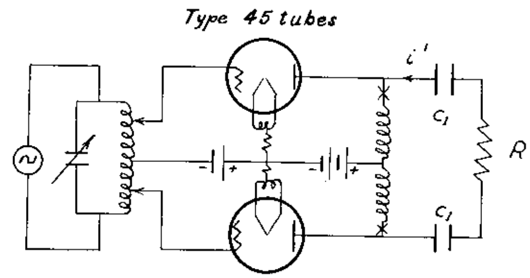


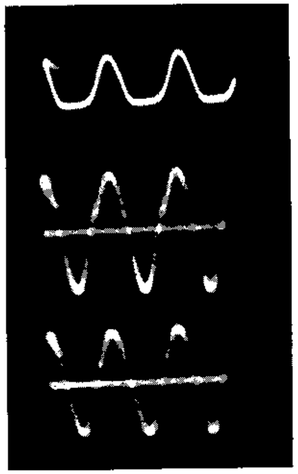
Fig. 8

(2) Characteristic kink in the plate voltage wave which not only causes the maximum plate voltage to be less than what would be expected from simple sine wave considerations but also distorts it very considerably in case the leakage reactance is too high; and

(3) Third harmonic distortion introduced into the load-current wave form.

Description of Experimental Setup and Oscillograms Obtained

To see in how far these theoretical curves are checked by actual experimental setups, the circuit shown in Fig. 8 was used. The input was derived from a class A amplifier of approximately two-watt capacity and the tuned circuit in the grid insures the input wave to be sinusoidal. The output choke was what one would use in the filter system of a sixty-cycle B battery eliminator. An RCA portable cathode-ray oscillograph, type TMV-122-B, was used and internally synchronized to obtain a stationary wave form on the fluorescent screen which was then photographed to nearly actual size by a camera with f 4.5 lens on timed exposure. In order to get the current wave through one tube, a small resistance (37.5 ohms) was introduced into each tube circuit at the ground end and two separate transformers were used to heat the filaments. This was necessary because one deflecting plate of the cath-



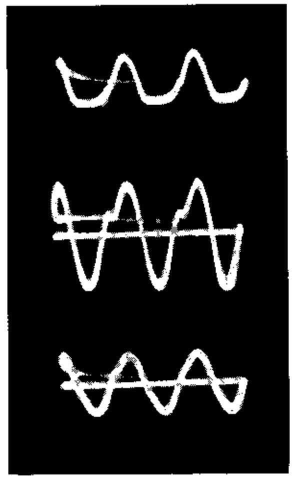


Fig. 9—$f=800$; $\omega L_1=250$; $r=1500$; $\rho=1500$; $m=6$; $n=24$.
Fig. 10—$f=800$; $\omega L_1=900$; $r=1500$; $\rho=1500$; $m=1.67$; $n=6.7$.

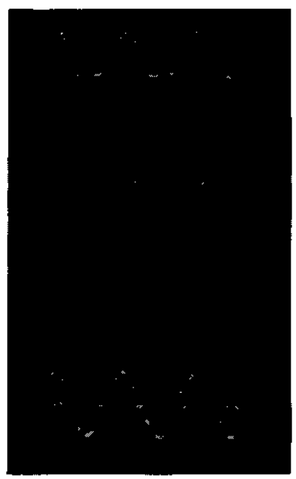


Fig. 11—$f=800$; $\omega L_1=250$; $r=750$; $\rho=1500$; $m=6$; $n=18$.
Fig. 12—$f=800$; $\omega L_1=900$; $r=750$; $\rho=1500$; $m=1.67$; $n=5$.

ode-ray tube had to be permanently grounded and could not be connected to any ungrounded resistance in the plate circuit. A positive crest voltmeter was connected into the grid circuit for the purpose of

adjusting the input exciting peak E to equal the C bias E_c. A positive trough voltmeter[7] was connected in the plate circuit to measure the minimum plate voltage, $e_{p\ \mathrm{min}}$. Because of the distorted plate voltage wave, it may be noted that $2(E_b-e_{p\ \mathrm{min}})$ does not equal $\sqrt{2}\,RI_{\mathrm{eff}}$, which holds only when the waves are sinusoidal.

The oscillograms shown in Figs. 9, 10, 11, and 12 indicated that the three main points noted on the theoretically calculated curves were checked quite satisfactorily. Of course, an exact quantitative agreement would not be expected because the tube characteristic could not be considered as linear and the cutoff was not as sharp as assumed in the linear characteristic. In order to show that the wave forms depend only on the ratio of the resistance to leakage reactance for frequencies not low enough to call into play the effect of the primary inductance, artificial leakage was introduced into the circuit by connecting two separate identical telephone repeater coils each in series with the choke at points shown by X in the diagram. Oscillograms substantiated the theory in showing that with $\omega L_1/\rho$ and ρ/r constant the wave shapes were nearly the same in spite of the fact that the frequency ratios were quite large.

Conclusion

From the above experimental results, it is evident that on the basis of the two simple assumptions, the wave shape in a class B audio amplifier can be calculated quite satisfactorily. Also the equivalent circuit may be depended upon to give correct results. In order not to draw the present paper to too lengthy a study, only the theoretical aspect of the solution of the problem has been touched. Such practical problems as what limiting values of leakage inductances may be tolerated without impairing the operating characteristics of the amplifier have not been analyzed. Another interesting problem would be to consider the effect of the primary inductance on the wave shape as the frequency becomes lower. A third practical problem would be to study the effect of dissymmetry in the transformer on the performance by making use of an equivalent circuit similar to Fig. 4. The other more difficult problems are to take into account the effects of grid current transients or to solve the problems without being restricted by the linear characteristic for the tubes.

Acknowledgment

The writer is greatly indebted to Dr. F. E. Terman and Mr. James Sharp of Stanford University for laboratory facilities and suggestions in carrying out the experiments.

[7] F. E. Terman, "Measurements in Radio Engineering," McGraw-Hill Book Company.

Mathematical Appendix

Fig. 2 shows the circuit. Let the symbols have the following significance:

μ = amplification factor of the tubes
ρ = plate resistance of one tube
E_b = direct voltage supplied by B source
E_c = magnitude of C bias voltage; $\mu E_c = E_b$
E = peak value of alternating voltage impressed on the grid $= E_c$
i_p, i_p' = plate currents through the upper and the lower tubes, respectively, at any time
e_p, e_p' = plate voltages on upper and lower tubes, respectively, at any time
e_g, e_g' = grid voltages on upper and lower tubes, respectively, at any time; $e_g = -E_c + E \cos \omega t$; $e_g' = -E_c - E \cos \omega t$
L = self-inductance of one leg of the primary of transformer with the secondary and the other leg both open-circuited
M_1 = mutual inductance between the two legs of the primary
$L_1 = L - M_1$ = leakage inductance of one leg of primary with respect to the other leg
L_b = self-inductance of the secondary with both legs of the primary open-circuited
M_2 = mutual inductance between each leg of primary and the secondary
$L_2 = L - M_2^2/L_b$ = total leakage inductance between one leg of primary and the secondary referred to the former
$N = \sqrt{L/L_b}$ = ratio of number of turns in one leg of primary to that in the secondary
r' = actual load resistance connected across the secondary
$r = N^2 r'$ = load resistance referred to one leg of the primary
$R = 4r$ = load resistance referred to the whole of primary; i.e., plate to plate
i' = actual current in the secondary or load resistance
i = load current referred to one leg of the primary $= i'/N$
$p = d/dt$ = differentiating operator
$\theta = \omega t$; $p\theta = \omega$

From Kirchhoff's laws, the plate voltages are:

$$e_p = E_b - Lpi_p + M_1 pi_p' + M_2 pi' \tag{16}$$

$$e_p' = E_b - Lpi_p' + M_1 pi_p - M_2 pi' \tag{17}$$

and for the secondary circuit we have

$$L_b pi' + r'i' - M_2 p(i_p - i_p') = 0 \tag{18}$$

or,

$$i' = \frac{M_2 p(i_p - i_p')}{L_b p + r'}. \tag{18a}$$

These three equations relate to the circuits external to the tubes. The linear characteristic of the tubes give:

$$\rho i_p = e_p + \mu e_g \quad \text{and} \quad \rho i_p' = e_p' + \mu e_g'. \tag{19}$$

Consider first the case when both tubes are conducting. Substituting i' from (18a) into (16) and (17) and then use e_p and e_p' in the characteristic (19), it will be found after simplification that

$$\rho i_p = \mu E \cos \theta - \frac{(L_2 p + r)Lp}{Lp + r} i_p + \frac{L(L_2 - L_1)p^2 + rM_1 p}{Lp + r} i_p' \tag{20}$$

and,

$$\rho i_p' = -\mu E \cos \theta - \frac{(L_2 p + r)Lp}{Lp + r} i_p' + \frac{L(L_2 - L_1)p^2 + rM_1 p}{Lp + r} i_p. \tag{21}$$

Adding (20) and (21) we get

$$\rho(i_p + i_p') + L_1 p(i_p + i_p') = 0 \tag{22}$$

giving therefore

$$i_p + i_p' = 2B\epsilon^{-m\theta} \tag{23}$$

where,

$$m = \rho/\omega L_1; \tag{24}$$

and $2B$ is a constant of integration. Substracting (21) from (20) gives:

$$\rho(i_p - i_p') = 2\mu E \cos \theta - \frac{L(2L_2 - L_1)p^2 + r(L + M_1)p}{Lp + r}(i_p - i_p'). \tag{25}$$

Assuming L and M_1 to be large and $L + M_1$ to be approximately $2L$, (25) becomes

$$[(\rho + 2r) + (2L_2 - L_1)p](i_p - i_p') = 2\mu E \cos \theta \tag{26}$$

from which the general solution is

$$i_p - i_p' = 2C\epsilon^{-s\theta} + \frac{2\mu E \cos(\theta - \cot^{-1} s)}{Z_s} \tag{27}$$

where,

$$s = (\rho + 2r)/\omega(2L_2 - L_1) \tag{28}$$

and,

$$Z_s^2 = (\rho + 2r)^2 + \omega^2(2L_2 - L_1)^2. \tag{29}$$

It may also be noted from (18a) that if L is large, then

$$i = \frac{i'}{N} = (i_p - i_p'). \tag{30}$$

In case the secondary is completely coupled to the primary, (the two legs of the primary are, however, not completely coupled), then

$$L_b = 2L + 2M_1 \quad \text{and} \quad M_2 = L + M_1 \tag{31}$$

so that

$$L_2 = L - M_2^2/L_b = (L - M_1)/2 = L_1/2. \tag{32}$$

The physical meaning of L_2 in this case is quite interesting. It represents the inductance measured between the center tap and the two outer terminals of the primary connected together, this inductance being the same as the parallel value of two identical uncoupled coils each having an inductance L_1. Thus when the load resistance is connected directly across the output circuit from plate to plate as shown in Fig. 8, $s=\infty$ and $\cot^{-1} s=0$, the transient term having time constant s does not exist and $Z_s=\rho+2r$, a nonreactive resistance. Combining the results, the complete expressions for i_p, i_p', e_p, e_p', and i for the period in which both i_p and i_p' are positive are

$$i_p = C\epsilon^{-s\theta} + B\epsilon^{-m\theta} + \frac{\mu E \cos(\theta - \cot^{-1} s)}{Z_s}$$

$$i_p' = -C\epsilon^{-s\theta} + B\epsilon^{-m\theta} - \frac{\mu E \cos(\theta - \cot^{-1} s)}{Z_s}$$

$$e_p = E_b + \rho C\epsilon^{-s\theta} + \rho B\epsilon^{-m\theta} - \frac{Z_1 \mu E \cos(\theta - \cot^{-1} s + \phi_1)}{Z_s}$$

where,

$$Z_1^2 = 4r^2 + \omega^2(2L_2 - L_1)^2$$

and,

$$\cos\phi_1 = 2r/Z_1$$

$$e_p' = E_b - \rho C\epsilon^{-s\theta} + \rho B\epsilon^{-m\theta} + \frac{Z_1 \mu E \cos(\theta - \cot^{-1} s + \phi_1)}{Z_s}$$

and,

$$i = 2C\epsilon^{-s\theta} + \frac{2\mu E \cos(\theta - \cot^{-1} s)}{Z_s}$$

which are the equations (3) given in the body of the paper.

Coming to the case when one tube is not conducting, e.g., $i_p'=0$, we find

$$\rho i_p = \mu E \cos\theta - \frac{(L_2 p + r)Lp}{Lp + r}\, i_p. \tag{33}$$

Assuming L to be large, this simplifies to

$$(\rho + r + L_2 p)i_p = \mu E \cos\theta \tag{34}$$

giving the complete solution

$$i_p = A\epsilon^{-n\theta} + \frac{\mu E \cos(\theta - \cot^{-1} n)}{Z_n} = i \tag{35a}$$

where,

$$n = (r + \rho)/\omega L_2$$

and,

$$Z_n^2 = (r + \rho)^2 + \omega^2 L_2^2$$

and A is another constant of integration. The values of e_p and e_p' are then

$$e_p = E_b + \rho A\epsilon^{-n\theta} - \frac{Z_2 \mu E \cos(\theta - \cot^{-1} n + \phi_2)}{Z_n} \tag{35b}$$

where,

$$Z_2^2 = r^2 + \omega^2 L_2^2; \quad \cos\phi_2 = r/Z_2;$$

and,

$$\begin{aligned} e_p' &= E_b + [r + (L_2 - L_1)p]i_p \\ &= E_b + \frac{(r + \rho)L_1 - \rho L_2}{L_2} A\epsilon^{-n\theta} \\ &\quad + \frac{Z_3 \mu E \cos(\theta - \cot^{-1} n + \phi_3)}{Z_n} \end{aligned} \tag{35c}$$

with $Z_3^2=r^2+\omega^2(L_2-L_1)^2$, and $\cos\phi_3=r/Z_3$. These are the equations (4) given in the body of the paper. As already explained, only two auxiliary angles α and β are needed to define the transitional points between the four periods. Denoting these periods by subscripts 1, 2, 3, 4, the set of equations (3) may be given a subscript 1, those in set (4) subscript 2, and then relations (6) and (7) are valid. Thus the continuity of the different quantities requires

$$i_{p1}(-\alpha) = 0 \quad \text{(defining equation for } \alpha) \tag{36a}$$

$$i_{p1}'(-\beta) = 0 \quad \text{(defining equation for } \beta) \tag{36b}$$

$$i_{p2}(\pi - \alpha) = i_{p3}(\pi - \alpha) = i_{p1}'(-\alpha) \tag{36c}$$

$$i_{p1}(-\beta) = i_{p2}(-\beta) \tag{36d}$$

and,

$$e_{p1}(2\pi - \alpha) + \mu e_{g4}(2\pi - \alpha) = 0. \tag{36e}$$

The above five equations are sufficient to determine the five quantities A, B, C, α, and β. Solving them simultaneously it is found that the three constants of integration may be expressed in terms of the two angles α and β as follows:

$$A = -\frac{L_2\epsilon^{m(\pi-\alpha)}}{(r+\rho)L_1 - \rho L_2}\frac{Z_m}{Z_n}\mu E \cos(\alpha - \cot^{-1} m + \cot^{-1} n) \tag{37a}$$

with,

$$Z_m^2 = \rho^2 + \omega^2 L_1^2$$

$$B = -\frac{L_1\epsilon^{-m\alpha}}{2[(r+\rho)L_1 - \rho L_2]}\mu E \cos\alpha \tag{37b}$$

$$C = -\frac{\mu E[\epsilon^{m\alpha}\cos(\beta + \cot^{-1} s) + \epsilon^{m\beta}\cos(\alpha + \cot^{-1} s)]}{Z_s[\epsilon^{s\alpha+m\beta} + \epsilon^{s\beta+m\alpha}]}. \tag{37c}$$

As for the values of α and β, they can be found from the following two additional equations after eliminating A and B:

$$2B\epsilon^{m\beta} = A\epsilon^{n\beta} + \frac{\mu E \cos(\beta + \cot^{-1} n)}{Z_n} \tag{37d}$$

and,

$$B = \frac{\mu E}{Z_s}\frac{[\epsilon^{s\alpha}\cos(\beta + \cot^{-1} s) - \epsilon^{s\beta}\cos(\alpha + \cot^{-1} s)]}{[\epsilon^{s\alpha+m\beta} + \epsilon^{s\beta+m\alpha}]}. \tag{37e}$$

When the secondary is completely coupled to the primary, then $2L_2 = L_1$, $s = \infty$, $\cot^{-1} s = 0$, $C = 0$, and B of (37e) simplifies to:

$$B = \frac{\mu E}{\rho + 2r}\left[\frac{\cos\beta - \cos\alpha}{\epsilon^{m\beta} + \epsilon^{m\alpha}}\right] \tag{38a}$$

and the value of A in (37a) becomes

$$A = -\frac{\epsilon^{m(\pi-\alpha)}}{r+\rho}\left[\frac{Z_m}{Z_n}\mu E \cos(\alpha - \cot^{-1} m + \cot^{-1} n)\right]. \tag{38b}$$

Combining (38a) and (37b) we get one relation which α and β must satisfy as follows:

$$\epsilon^{-m\alpha}\cos\alpha = -\epsilon^{-m\beta}\cos\beta. \tag{39}$$

Substituting the values of A and B from (38b) and (38a) into (37d) and simplifying, the second relation that α and β must satisfy is

$$\epsilon^{-n(\pi+\beta-\alpha)} = -\frac{(1+mn)\cos\alpha + (n-m)\sin\alpha}{(1+mn)\cos\beta + (n-m)\sin\beta}. \tag{40}$$

To evaluate α and β from the two transcendental equations (39) and (40) is not as difficult as it might appear at first sight, because α and β under ordinary circuit arrangements are both nearly equal to $\pi/2$ and n is fairly large, so that the value of α is approximately obtained by setting the numerator of (40) to zero; i.e.,

$$\tan\alpha \leqq -\frac{1+mn}{n-m} \tag{41}$$

or,

$$\alpha \geqq \tan^{-1}\frac{1+mn}{m-n}. \tag{42}$$

With the value of α so found, the corresponding value of β can be easily obtained from (39) with the help of logarithmic and trigonometric tables. A second and closer approximation can be obtained by checking up the relation of continuity

$$e_{p2}'(-\beta) = e_{p1}'(-\beta). \tag{43}$$

This relation is chosen for this purpose in preference to the others be-because a slight error in α and β shows up most prominently in the inequality of both sides of (43). If this relation does not check to within a small fraction of a per cent, a slightly different value for α should be used and the corresponding β found, and the same equality checked again. Usually two or three trials are sufficient to yield results that are accurate enough for curve plotting.